CH NN – THE REAL NUMBERS AND THEIR PROPERTIES

You have seen many different kinds of numbers in your algebra studies:

7	0	-9	2.835	$-\frac{15}{4}$
$\frac{1}{3}$	$\frac{23}{75}$	π	$-\sqrt{2}$	$2.7383838\ldots$

$12.09358573008628\ldots$ $\qquad$ $-7.03003000300003\ldots$

As different as all these numbers may seem from each other, they actually share one critical common characteristic: They can all be written as **decimal numbers**.

❑ *THE REAL NUMBERS*

Here are all the numbers in the Introduction written as decimals:

$7 = 7.0$	A finite decimal
$0 = 0.0$	A finite decimal
$-9 = -9.0$	A finite decimal
2.835	A finite decimal

> Actually, all the finite decimals listed here can be viewed as infinite, repeating decimals. How? By realizing that each finite decimal could have an unending supply of zeros attached to their back end. Thus, any finite decimal is really a repeating decimal.

$-\frac{15}{4} = -3.75$	A finite decimal
$\frac{1}{3} = 0.33333\ldots$	An infinite, repeating decimal
$\frac{23}{75} = 0.2787878\ldots$	An infinite, repeating decimal (the 78)
$\pi = 3.14159265\ldots$	An infinite, non-repeating decimal
$-\sqrt{2} = -1.41421356\ldots$	An infinite, non-repeating decimal
$12.09358573008628\ldots$	An infinite, non-repeating decimal
$-7.03003000300003\ldots$	An infinite, non-repeating decimal

Note that the first seven decimals listed above repeat a block of digits forever (the ***rational*** numbers). In contrast, the last four don't repeat (and are called ***irrational*** numbers). Nevertheless, they are all decimals.

But what about a number like $\sqrt{-9}$? This number must be a number whose square is –9. Now, what number do we know which, when squared, would come out –9?

Does 3 work? No, since $3^2 = 9$. Does –3 work? No, since $(-3)^2 = 9$, as well.

F.Y.I.

π is the ratio of the circumference of any circle to its diameter.

$\sqrt{2}$ is the length of the hypotenuse of a right triangle where each leg has length 1.

We conclude that no decimal in the world can represent the number $\sqrt{-9}$, or for that matter, the square root of any negative number.

To distinguish between the numbers that are decimals and numbers like $\sqrt{-9}$, which can never be written as a decimal, the term **real number** was given to the decimals, and the term **imaginary number** was given to numbers like $\sqrt{-9}$.

Hundreds of years ago, mathematicians thought it was obvious which numbers were real and which were imaginary. But this demonstrates a rather arrogant attitude. After all, to a beginning algebra student, a real number like $\sqrt{2}$ (which is an infinite, non-repeating decimal) may not seem very "real" at all. Moreover, imaginary numbers, like $\sqrt{-1}$, seem very real to people in fields such as electronics and quantum physics, who use them every day. The bottom line is that the terms *real* and *imaginary* are entirely arbitrary — one person's reality is another's imagination. But we're stuck with these terms, so we might as well learn them.

The REAL NUMBERS, $\mathbb{R}$, can be divided into two disjoint subsets:

The Rationals – the repeating decimals (which includes the finite decimals).

The Irrationals – the decimals that are non-repeating.

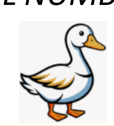
In summary, we call any number that can be written as a decimal a ***real number***. The set of real numbers is often denoted by writing $\mathbb{R}$. As far as the first courses in Algebra are concerned, the only numbers you will come across that won't be real numbers are the square roots, the fourth roots, the sixth roots, and so on, of negative numbers.

Homework

1. Classify each number as **real** or **imaginary**:

a. 123	b. -42	c. 0
d. 2.3	e. $\sqrt{-8}$	f. $\sqrt{144}$
g. $-\sqrt{81}$	h. $\sqrt{10}$	i. -23.78
j. $-\pi$	k. $\sqrt{3}$	l. $\sqrt{-121}$
m. 0.239057	n. 2.787878. . .	o. 3.092748526
p. 3.1428669. . .	q. $\sqrt{8765}$	r. $\sqrt{-0.25}$
s. $-\sqrt{-25}$	t. $\sqrt{-(-71)}$	u. 10^6

If it looks like a decimal, swims like a decimal, and quacks like a decimal, IT'S A *REAL NUMBER*!



❑ THE CLOSURE PROPERTIES

Now that we know what a real number is, we can analyze some of the properties of these real numbers. We are quite familiar with all of these properties; it's the official names of these properties that might cause us some grief.

When a collie mates with a collie, it's quite certain that the offspring will be a collie. The same idea holds in the real numbers. For example, when two real numbers are added, the sum is a real number. Some examples:

17 + 14.56 = 31.56, which is a real number.

Let's add the real numbers π and $\sqrt{2}$. The sum is $\pi+\sqrt{2}$, which cannot be simplified, but I guarantee that the sum is a real number. Here's a little proof: Recall that a real number is any number which can be written as a decimal, so

$$\pi+\sqrt{2} = 3.1415926\ldots + 1.4142135\ldots = 4.5558062\ldots$$

which is certainly a decimal, and therefore a real number. Hence $\pi+\sqrt{2}$ is a real number.

Because of this property — that adding real numbers always results in a real number — we say that the set of real numbers is ***closed*** under addition, The set of real numbers is also closed under subtraction and multiplication. And with one critical exception, it's also closed under division. Do you know what that exception is?

❑ THE COMMUTATIVE PROPERTIES

Just as a governor can commute your death sentence to a life sentence by "switching" things around, we can switch, or reverse, the factors of 7×3 and write it as 3×7, and we've known since we were little tykes that they're the same product, 21.

More generally, the factors of the product xy can be commuted, or switched, and be written yx, without changing the answer. We say that multiplication is a ***commutative*** operation. Anytime we want to switch the order of two factors, we can simply do it.

Addition is also a commutative operation; after all, it's pretty clear that $x + y = y + x$ for any real numbers x and y.

Now it gets interesting: Is subtraction a commutative operation? That is, does $x - y = y - x$ for all values of x and y? Of course not; does $10 - 2 = 2 - 10$?

You decide whether division is a commutative operation.

In summary, the ***commutative properties*** state the following: For any real numbers x, y, and z:

$$\boldsymbol{x + y = y + x}$$
$$\boldsymbol{xy = yx}$$

❑ THE ASSOCIATIVE PROPERTIES

Consider the sum $5 + 3 + 2$. We've learned via the *Order of Operations* chapter that we can certainly work the problem left to right, which means we're going to add the 5 and 3 first; let's use parentheses around the $5 + 3$:

$(5 + 3) + 2 = 8 + 2 = \mathbf{10}$ Here we "associate" the 5 and 3.

Now let's see if we get the same answer if we add the 3 and 2 first:

$5 + (3 + 2) = 5 + 5 = \mathbf{10}$ Here we "associate" the 3 and 2.

We get the same answer. It appears that $(5 + 3) + 2 = 5 + (3 + 2)$. That is, we can "shift" the parentheses left-to-right or right-to-left.

The same property holds for multiplication, too. For example,

$(7 \times 5) \times 2 = 35 \times 2 = \mathbf{70}$

and, $7 \times (5 \times 2) = 7 \times 10 = \mathbf{70}$

In summary, the ***associative properties*** state the following: For any real numbers x, y, and z:

$$(x + y) + z = x + (y + z)$$
$$(xy)z = x(yz)$$

❑ THE DISTRIBUTIVE PROPERTY

This property is the cornerstone of algebra, and is utilized constantly throughout all branches of mathematics. It is the only property in this chapter which involves two different operations, multiplication and addition.

For any real numbers a, b, and c:

$$a(b + c) = ab + ac$$

❑ THE ADDITIVE IDENTITY

Would you agree that adding **0** to a real number results in that same number? In other words, $\boldsymbol{x + 0 = x}$ for all real numbers x. Of course you would agree; you knew that when you were a little kid. Leave it to the math geeks to come up with a fancy term for something so simple.

We call the number zero the ***additive identity*** because when zero is **added** to a number, the sum is **identical** to that number. There is only one additive identity in the set of real numbers. In other words, zero is the only real number that acts like zero.

❑ THE MULTIPLICATIVE IDENTITY

More silliness — we know that any number times **1** is itself: $\boldsymbol{x \cdot 1 = x}$ for any real number x. This special number, 1, is given the name ***multiplicative identity*** (accent on the "plic"), because when a number is **multiplied** by 1, the product is **identical** to that number. And it should be clear that 1 is the only number that possesses this property.

❑ THE ADDITIVE INVERSE

Remember the term "opposite"? The opposite of 7 is –7, the opposite of –99 is 99, and the opposite of 0 is 0. The official term for "opposite" is ***additive inverse***, so now we can talk fancy and say things like, "The additive inverse of –23 is 23." Every real number has an additive inverse; in general, the additive inverse of n is $-n$. Also notice that when a number and its additive inverse are added, the sum is always 0, the additive identity. That is, for every real number n,

$$\boldsymbol{n + (-n) = 0}$$

❑ THE MULTIPLICATIVE INVERSE

Here's some more jargon: The phrase *multiplicative inverse* simply means "reciprocal." Thus, the multiplicative inverse of $\frac{3}{5}$ is $\frac{5}{3}$. In general, the multiplicative inverse of x is $\frac{1}{x}$, and the multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$. Does every real number have a multiplicative inverse? Absolutely NOT. Consider zero; if zero had a multiplicative inverse, it would be $\frac{1}{0}$, which is undefined. Therefore, every real number except zero has a multiplicative inverse. Notice that when a number and its multiplicative inverse are multiplied, the product is always 1, the multiplicative identity. In other words, for any real number x (other than 0),

$$\boldsymbol{x\left(\frac{1}{x}\right) = 1}$$

Homework

2. a. The set of real numbers, $\mathbb{R}$, is **closed** under division, with one exception. What is that exception?

   b. Explain what it means to say that the set of real numbers is closed under multiplication.

3. Which operations (addition, subtraction, multiplication, division) are **commutative** operations in $\mathbb{R}$? Give an example which demonstrates which are commutative and which are not.

4. Prove that subtraction and division are NOT **associative** operations in $\mathbb{R}$.

5. What's the official term for the number **0** in $\mathbb{R}$?

6. What's the official term for the number **1** in $\mathbb{R}$?

7. Find the **additive inverse** of each real number:

   a. 17 b. -3 c. 0 d. $-\pi$
   e. $-\frac{7}{4}$ f. x g. $-n$

8. The <u>sum</u> of any real number and its additive inverse is ___.

9. Find the **multiplicative inverse** of each real number:

a. 22 b. -9 c. 0

d. $\frac{1}{\pi}$ e. $-\frac{\sqrt{3}}{\sqrt{2}}$ f. N

10. The <u>product</u> of any real number and its multiplicative inverse is ___.

11. a. T/F: Every real number has an opposite.

b. T/F: Every real number has a reciprocal.

MATCHING:

12. __	$a(b + c) = ab + ac$	a. commutative
13. __	$a + b = b + a$	b. associative
14. __	xy is a real number if x and y are.	c. distributive
15. __	$-a$	d. closure
16. __	1	e. additive identity
17. __	$\frac{1}{a}$	f. additive inverse of a
18. __	$x(yz) = (xy)z$	g. multiplicative identity
19. __	0	h. multiplicative inverse of a
20. __	$uw = wu$	
21. __	$(a + b) + c = a + (b + c)$	
22. __	$a - b$ is a real number if a and b are real numbers	
23. __	$x + (y + z) = x + (z + y)$	
24. __	$a^2b^3 = b^3a^2$	

25. __ If w is a real number, so is w^6.

26. __ $\sqrt{2} - \sqrt{3} + 10\pi$ is a real number.

27. __ $(ab)(cd) = a(bc)d$

Solutions

1.

a.	Real	b.	Real	c.	Real	d.	Real
e.	Imaginary	f.	Real	g.	Real	h.	Real
i.	Real	j.	Real	k.	Real	l.	Imaginary
m.	Real	n.	Real	o.	Real	p.	Real
q.	Real	r.	Imaginary	s.	Imaginary	t.	Real
u.	Real						

2. a. Dividing any number by 0 does <u>not</u> produce a real number.
b. It means that if x and y are any two real numbers, their product xy is also a real number. For example, since π and $\sqrt{7}$ are real numbers, so is their product $\pi\sqrt{7}$.

3. Addition and multiplication are commutative operations. For example,

$$1 + \sqrt{2} = \sqrt{2} + 1 \quad \text{and} \quad 100 \times \pi = \pi \times 100$$

Subtraction and division are NOT commutative operations. For instance,

$$34 - 10 \neq 10 - 34 \quad \text{and} \quad \frac{10}{2} \neq \frac{2}{10}$$

(Confirm these statements!)

4. If subtraction were associative, the following would have to be true:

$$10 - (7 - 4) = (10 - 7) - 4$$

$$\Rightarrow \quad 10 - 3 = 3 - 4$$

$$\Rightarrow \quad 7 = -1$$, which, of course, it isn't.

If division were associative, the following would hold:

$$20 \div (10 \div 2) = (20 \div 10) \div 2$$

The problem is, the left side equals 4, while the right side equals 1.

5. Additive identity

6. Multiplicative identity

7. a. -17 b. 3 c. 0 d. π e. $\frac{7}{4}$ f. $-x$ g. n

8. 0

9. a. $\frac{1}{22}$ b. $-\frac{1}{9}$ c. Does not exist d. π e. $-\frac{\sqrt{2}}{\sqrt{3}}$ f. $\frac{1}{N}$

10. 1

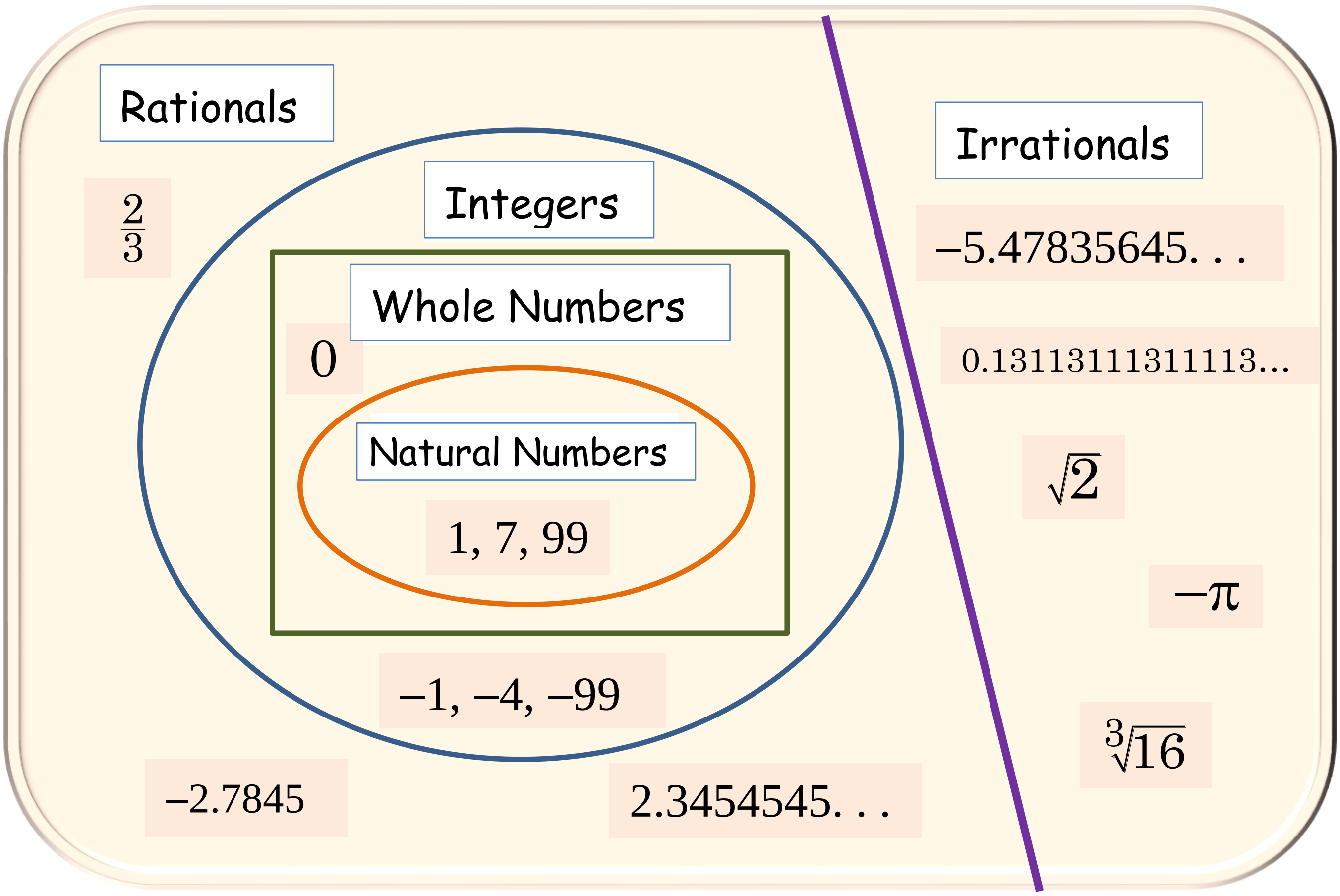
11. a. True b. False

12. c. **13.** a. **14.** d. **15.** f. **16.** g. **17.** h.

18. b. **19.** e. **20.** a. **21.** b. **22.** d. **23.** a.

24. a. **25.** d. **26.** d. **27.** b.

The Set of Real Numbers



– Every natural number is a whole number.

– Every whole number is an integer.

Therefore, every natural number is an integer.

– The only whole number that is not a natural number is 0.

– Every integer is rational.

Therefore, every natural number is rational,

and every whole number is rational.

– The rationals can be described as all repeating decimals.

– The irrationals can be described as all <u>non-repeating</u> decimals.

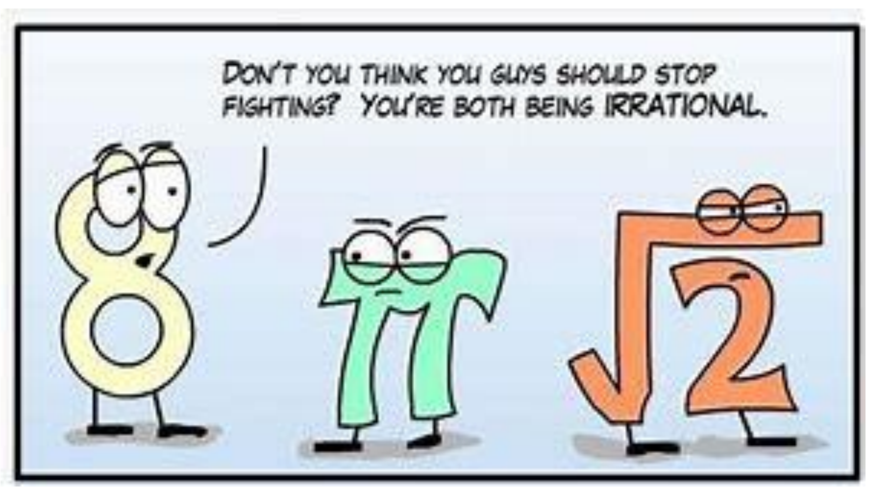
– The rationals and irrationals are *disjoint*; they have nothing in common:

Rationals $\cap$ Irrationals = $\varnothing$ [the null, or empty, set]

– The rationals and irrationals make up the real numbers.

Rationals $\cup$ Irrationals = Real Numbers

See Sets for more info on $\cap$, $\cup$, and $\varnothing$.



"It is harder
to crack a prejudice
than an atom."

– A. Einstein